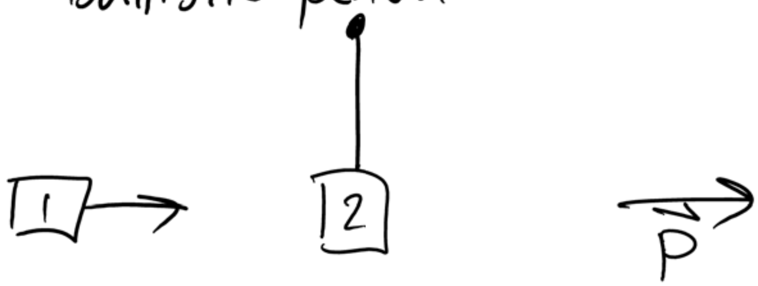
PHYS 410 - Sept 6

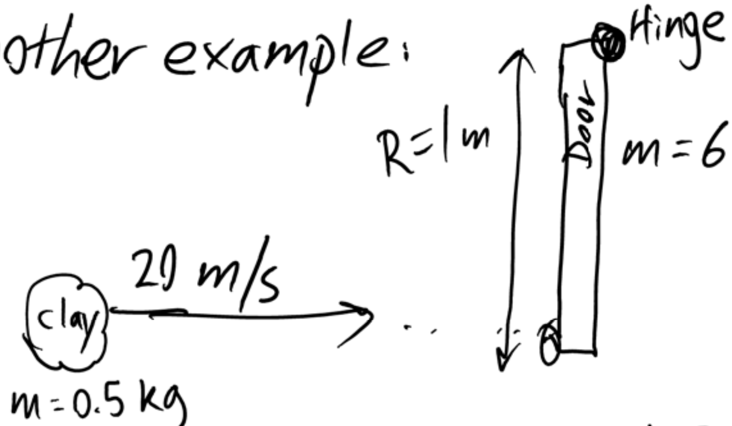
pdf of Tuesday lecture: ter.ps/phys410sep04
From today's lecture: ter.ps/phys410sep06

What's conserved in a collision?

Last time: Ballistic pendulum




Another example:




Clay sticks to door

Q: How fast will the door swing after it's hit?

What will be conserved? Not energy.

Fixed Pivot $\Rightarrow$ angular momentum will be conserved

Angular momentum (around the hinge)

Before collision: $\vec{L} = \vec{r} \times \vec{p} \Rightarrow |\vec{L}| = rp\sin\phi$

$= r_\perp p$

$= (1\,m) \times (10\,kg\,m/s)$

$= 10\,kg \cdot m^2/s$

After collision: $L = I_{total}\,\omega$

$$I_{total} = I_{clay} + I_{door}$$

$$= m_{clay}R^2 + \frac{1}{3}m_{door}R^2 = 2.5\ \text{kg}\cdot\text{m}^2$$

Aside: $I_{door} = \int dm\, r^2$


$$= \int_0^R \left(\frac{dr}{R}m_{door}\right) r^2$$

$$= \frac{m_{door}}{R}\,\frac{1}{3}R^3$$

$$\Rightarrow \omega = \frac{L}{I_{total}} = \frac{10\ \text{kg}\cdot\text{m}^2/\text{s}}{2.5\ \text{kg}\cdot\text{m}^2} = 4\ \text{s}^{-1} = 4\ \text{rad/s}$$

Check: Was linear momentum conserved?

Before: $p = 10\ \text{kg}\cdot\text{m/s}$

Just After: $p = p_{clay} + p_{door\,cm}$

$$= m_{clay}(\omega R) + m_{door}\left(\omega \frac{R}{2}\right)$$

$$= (0.5\,\text{kg})(4\,\text{rad/s})(1\,\text{m}) + (6\,\text{kg})(4\,\text{rad/s})(0.5\,\text{m})$$

$$= 2 + 12 = 14\ \text{kg}\cdot\text{m/s}$$

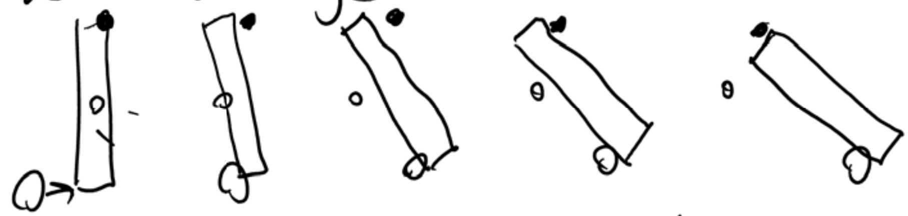
$\Rightarrow \vec{p}$ is not conserved in this collision

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = 4\ \text{kg}\cdot\text{m/s}\ \hat{x}$$

So the <u>impulse</u> was $4\ \text{kg}\cdot\text{m/s}\ \hat{x}$

$$= \int \vec{F}\, dt$$

If there had been no hinge:



$P_f = 10$ kg·m/s in this case

Bat demo:

Center of percussion

## Hamilton's principle

The path that a dynamical system actually takes from one point to another over a given time interval is one that makes the action integral stationary.

Action integral: $$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

Path: $x(t)$, i.e. position vs. time

points: starting and ending points of the path

$(t_1, x_1) \rightsquigarrow (t_2, x_2)$

with the endpoints fixed

Note: $S$ is a scalar, calculated from a function (the path) that can vary

So this action integral is a "functional"

Stationary: small variations in the path do not change $S$ to first order

{minimum, maximum, inflection point} → stationary

→ Solving this with the right $\mathcal{L}$ will tell us how the physical system behaves!

Use this... find the path which makes it stationary

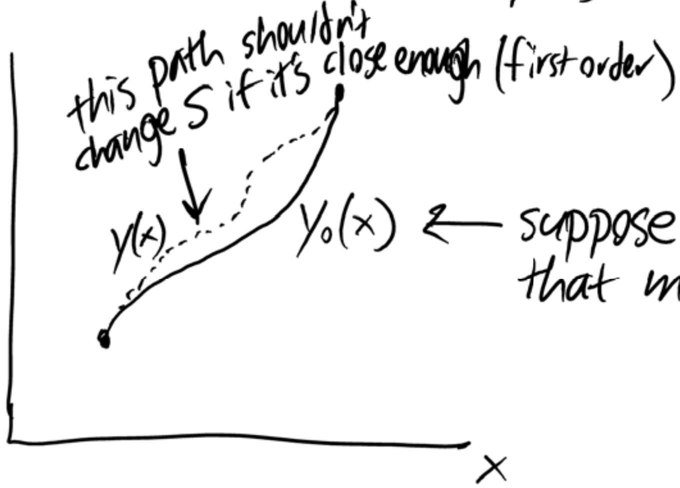
Calc. of variations

We have $S = \int_{x_1}^{x_2} f(y, y', x)\, dx$

$y(x)$ is the path

$y'(x) = \frac{dy}{dx}$

we want to find the $y(x)$ which makes this stationary




← suppose this is the path that makes $S$ stationary

write as $y(x) = y_0(x) + \alpha\,\eta(x)$

$\eta(x)$ is not small but $\alpha \to 0$

How does $S$ change from this?

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \frac{df}{d\alpha}\, dx = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} + \underbrace{\cancel{\frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}}}_{0} \right) dx$$

$$= \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x) \right) dx$$

use I.B.P. $\int v\,du = uv\Big| - \int u\,dv$

$$\int_{x_1}^{x_2} \underbrace{\frac{\partial f}{\partial y'}}_{\text{"}v\text{"}} \underbrace{\eta'(x)\,dx}_{\text{"}du\text{"}} = \underbrace{\left[\eta(x)\frac{\partial f}{\partial y'}\right]\Bigg|_{x_1}^{x_2}}_{\substack{=0 \text{ since} \\ \eta(x_1)=\eta(x_2)=0}} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$

$$\Rightarrow \frac{dS}{d\alpha} = \int_{x_1}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\,dx$$

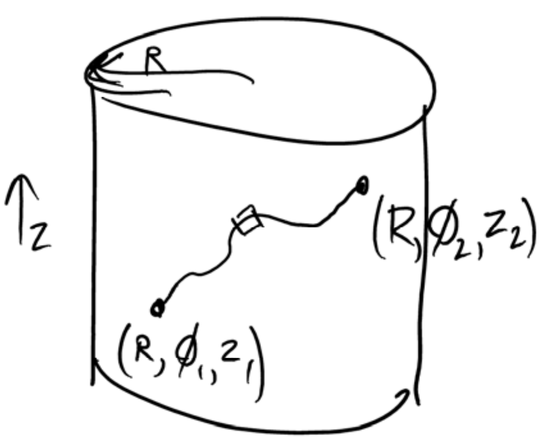
want this to be zero for <u>any</u> $\eta(x)$

$$\Rightarrow \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

This is the Euler-Lagrange equation

Means: The $y(x)$ [path] which satisfies this equation makes $S$ stationary

Example: Shortest path between two points on a cylinder

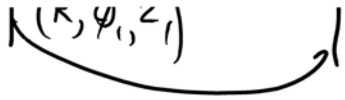



Length of small piece

$$ds = \sqrt{dz^2 + (R\,d\phi)^2}$$

suppose that $\phi$ is a function of $z$

$$\Rightarrow ds = \sqrt{1 + R^2\left(\frac{d\phi}{dz}\right)^2}\,dz$$



$$\Rightarrow ds = \sqrt{1 + R^2\left(\frac{d\phi}{dz}\right)^2}\, dz$$

Total length: $L = \int ds = \int_{z_1}^{z_2} \sqrt{1+R^2\phi'^2}\, dz$

want to minimize $L$ by varying $\phi(z)$

use E-L equation with $f = \sqrt{1+R^2\phi'^2}$

$$\frac{\partial f}{\partial \phi} = \frac{d}{dz}\left(\frac{\partial f}{\partial \phi'}\right)$$

$$0 = \frac{d}{dz}\left(\frac{1}{2}\left(1+R^2\phi'^2\right)^{-1/2} \cdot R^2 \cdot 2\phi'\right)$$

$$\Rightarrow \frac{R^2\phi'}{\sqrt{1+R^2\phi'^2}} = C$$

$$R^4\phi'^2 = C^2\left(1+R^2\phi'^2\right) \longrightarrow \phi'^2 = \frac{C^2}{\underbrace{R^4 - C^2R^2}_{\text{a constant - call that } D}}$$

$\Rightarrow \phi'(z) = \sqrt{D}$ , a constant

$\Rightarrow \phi(z) = \phi_0 + \sqrt{D}\, z$

Impose fixed endpoints

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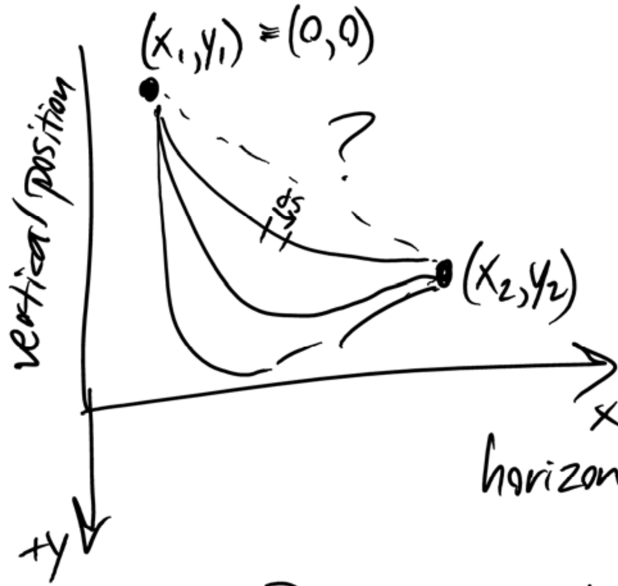
This was for a 1-D path, i.e. $y(x)$ or $\phi(z)$

In 2 or more dimensions, get an E-L equation for <u>each</u> of the coordinates

Brachistochrone




Given starting & ending points, find the track for a object to slide along that minimizes the travel time

Describe that path $x(y)$

"S" functional is the total time

$$t = \int_{path} dt = \int_{path} \frac{ds}{v}$$

$\leftarrow$ speed there

use Energy conservation:

$$mgy = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2gy}$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\; dy$$

$$t = \int \underbrace{\frac{\sqrt{x'^2+1}}{\sqrt{2gy}}}_{\text{this is "}f(x,x',y)\text{"}} dy$$

Use E-L equation:

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right)$$

$$\overset{\|}{0} \quad \Rightarrow \quad \frac{\partial f}{\partial x'} \text{ is constant} = C$$

$$\Rightarrow \frac{\frac{1}{2}\left(x'^2+1\right)^{-1/2}\cdot 2x'}{\sqrt{2gy}} = C \quad \Rightarrow \frac{x'}{\sqrt{x'^2+1}} = \underbrace{\sqrt{2g}\,C}_{c'}\sqrt{y}$$

$$\Rightarrow x'^2 = c'^2 y\left(x'^2+1\right)$$

$$x'^2\left(1-c'^2y\right) = c'^2 y \quad \Rightarrow x' = \sqrt{\frac{c'^2 y}{1-c'^2 y}}$$

Need to solve this D.E. to get our $x(y)$

Integrate... how?

Taylor: $y = a(1-\cos\theta) = \frac{1}{2c'^2}(1-\cos\theta)$

$$\Rightarrow \quad x' = \sqrt{\frac{\frac{1}{2}(1-\cos\theta)}{1-\frac{1}{2}(1-\cos\theta)}} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

Mult. by $\sqrt{\frac{1-\cos\theta}{1-\cos\theta}}$ $\Rightarrow$ $= \frac{1-\cos\theta}{\sin\theta}$

Also $dy = \frac{1}{2c'^2}\sin\theta\, d\theta$

$$\Rightarrow \quad x = \int \frac{1-\cos\theta}{\cancel{\sin\theta}}\cdot\frac{1}{2c'^2}\cancel{\sin\theta}\, d\theta$$

$\Rightarrow$ Get solution $x = \frac{1}{2c'^2}(\theta - \sin\theta) = a(\theta-\sin\theta)$

$$y = \qquad\qquad = a(1-\cos\theta)$$

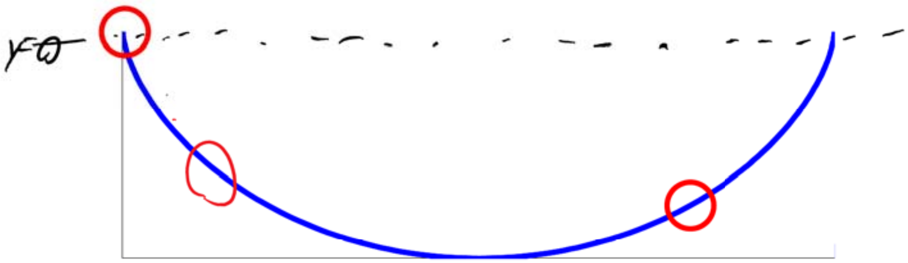
Parametric solution with parameter $\theta$
($x$ no longer has to be a single-valued function of $y$)

One adjustable parameter, $a$, which we must set to satisfy the boundary condition

i.e., ends at $(x_2, y_2)$

# Brachistochrone solution

Thursday, September 06, 2012
7:48 AM




cycloid

There is sometimes a unique solution,
and sometimes multiple solutions, depending on $(x_2, y_2)$

Comments:

For small $\theta$:

$$y = a(1-\cos\theta) \sim \frac{a}{2}\theta^2$$

$$x = a(\theta - \sin\theta) \sim \frac{a}{6}\theta^3$$

- Slope $\sim \frac{y}{x}$ is $\sim \frac{1}{3\theta}$ — infinite for small $\theta$, i.e. it departs vertically downward at first

- $$t = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{x'^2+1}}{\sqrt{y}}\, dy$$

$$x' = \frac{dx}{dy} = \frac{dx/d\theta}{dy/d\theta} = \frac{1-\cos\theta}{\sin\theta}$$

$\vdots$

$$= \frac{1}{\sqrt{2a}} \int^{\theta_f} \sqrt{2a}\, d\theta$$

$$= \frac{1}{\sqrt{2g}} \int_0^{\theta_f} \sqrt{2a}\, d\theta$$

$$= \sqrt{\frac{a}{g}}\, \theta_f$$