PHYS 410 - Sept 4

Tuesday, September 04, 2012
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Energy conservation examples

Total energy:

$$K_i + U_i = K_f + U_f$$

$$0 + mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 + 0$$

$I = \frac{1}{2}mR^2$ for a <u>solid</u> cylinder

$\omega = \frac{v}{R}$

Aside: $K_{tot} = \sum_i \frac{1}{2} m_i v_i^2$ in general

(bits of the object)

For a rigid object, separate out CM translational motion from the rotation of bits around the CM rotation of bits around the CM

$$\longrightarrow K_{tot} = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I \omega^2$$

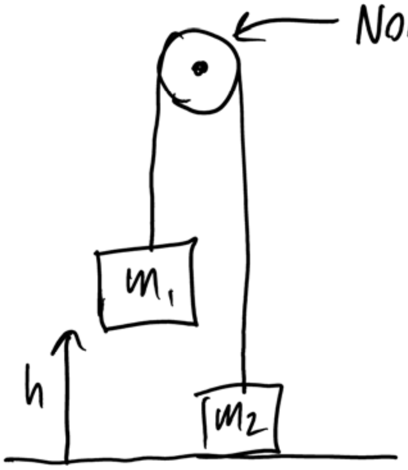
with $I = \sum_i m_i r_{\substack{\text{from}\\\text{axis}}}^2$

$$\rightarrow \cancel{m}gh = \frac{1}{2} mv^2 + \frac{1}{2}\left(\frac{1}{2} mv^2\right) = \frac{3}{4}\cancel{m}v^2$$

$$\Rightarrow v = \sqrt{\frac{4}{3} gh}$$

Atwood Machine

Non-negligible I



Q: How fast is mass 1 moving when it hits the ground?

$$0 + m_1 gh = \frac{1}{2}(m_1+m_2)v^2 + \underbrace{\frac{1}{2} I\omega^2}_{\frac{1}{2}\left(\frac{I}{R^2}\right)v^2} + m_2 gh$$

$$(m_1 - m_2)gh = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right) v^2$$

$$\Rightarrow v = \sqrt{\ \sim\ }$$

Follow-up: How much higher does mass 2 go?

$$\frac{1}{2} m_2 v^2 = m_2 g h_{additional}$$

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$$\Rightarrow h_{add} = \frac{v^2}{2g} = \frac{(m_1 - m_2)\, h}{m_1 + m_2 + I/R^2}$$

Notes: $h_{add} \propto h$ (proportional)

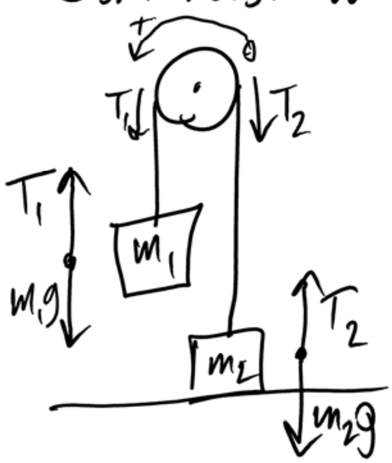
Limiting case: $m_1 \gg m_2$, $m_1 \gg I/R^2$

$$\Rightarrow h_{add} \approx \frac{m_1 h}{m_1} \approx h$$

Limiting case: $\frac{I}{R^2} \gg m_1, m_2$

$$\Rightarrow h_{add} \to 0$$

Contrast with using Newton's laws



$$\tau_{net} = RT_1 - RT_2 = I\alpha = I\frac{a}{R}$$

$$m_1 g - T_1 = m_1 a$$

$$T_2 - m_2 g = m_2 a$$

Combine 3 eqns, solve for things

$\Rightarrow$ same answer for v

& get T_1 & T_2

(forces which guarantee the constraints on string length & not slipping)

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Work-KE theorem: — limited usefulness!

$$\Delta K = W_{net,on}$$

on the system

Net work

$$W = \int \vec{F} \cdot d\vec{l}$$

Consider potential energy too —
A function of the configuration of the system, parametrized in this ideal case by the length of a Slinky, for example

Conservative forces $\Rightarrow$ can define a potential energy

$$\rightarrow \quad U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}) \cdot d\vec{r}$$

U is defined to be zero here

Minus sign because
$\vec{F}$ is force provided by the system, not on it

same $U(\vec{r})$ from equal integrals

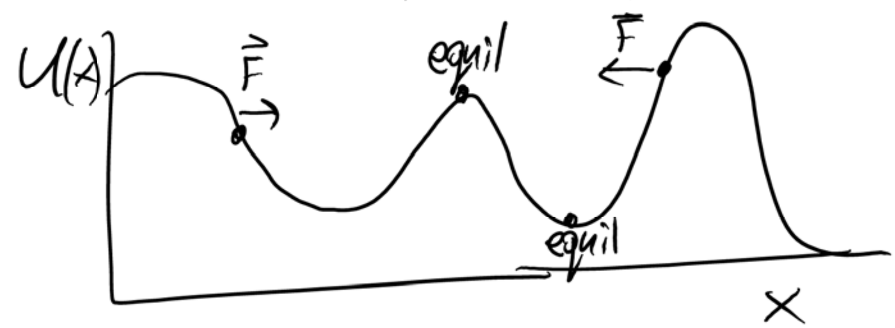
same $U(\vec{r})$ from equal integrals

Force associated with the potential energy:

$$F(x) = -\frac{dU}{dx} \text{ in 1-D}, \quad -\vec{\nabla}\underbrace{U(\vec{r})}_{\text{gradient is "uphill"}} \text{ in 3-D}$$

negative sign $\Rightarrow$ force is "downhill"

1-D picture:



In 3-D, you get 3 force components from this single function $U(\vec{r}) \equiv U(x,y,z)$

i.e. get F_x, F_y, F_z

What about if there are 2 particles?

$\rightarrow U(\vec{r}_1, \vec{r}_2)$ e.g. Coulomb ~~force~~ potential $\dfrac{kq_1q_2}{|\vec{r}_1 - \vec{r}_2|}$

There's a force on each charge

$$F_{on\,1} = -\vec{\nabla}_1 U$$

I mean $\dfrac{\partial}{\partial x_1}\hat{x}_1 + \dfrac{\partial}{\partial y_1}\hat{y}_1 + \dfrac{\partial}{\partial z_1}\hat{z}_1$

$$F_{\text{on }2} = -\vec{\nabla}_2 U$$

Just an extension of (x, y, z)

$\Rightarrow$ 6 force components

$$F_{x_1},\ F_{y_1},\ F_{z_1},\ F_{x_2},\ F_{y_2},\ F_{z_2}$$

could be 6 arbitrary unrelated coordinates

$\Rightarrow$ Generalized "gradient" of $U(\ )$ simultaneously gives forces on <u>all</u> degrees of freedom

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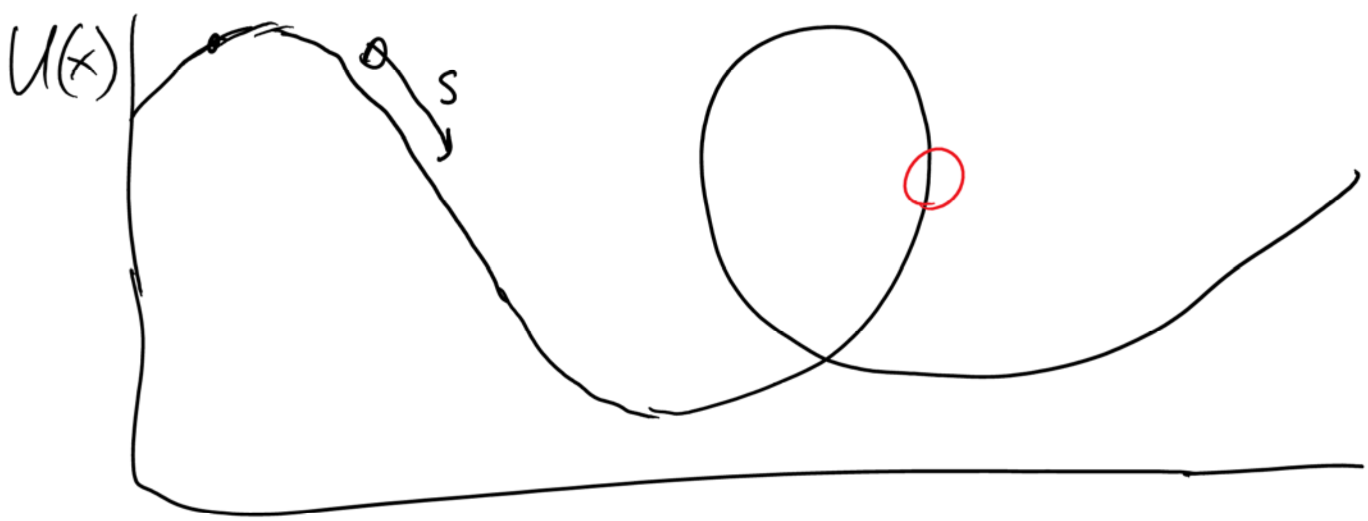
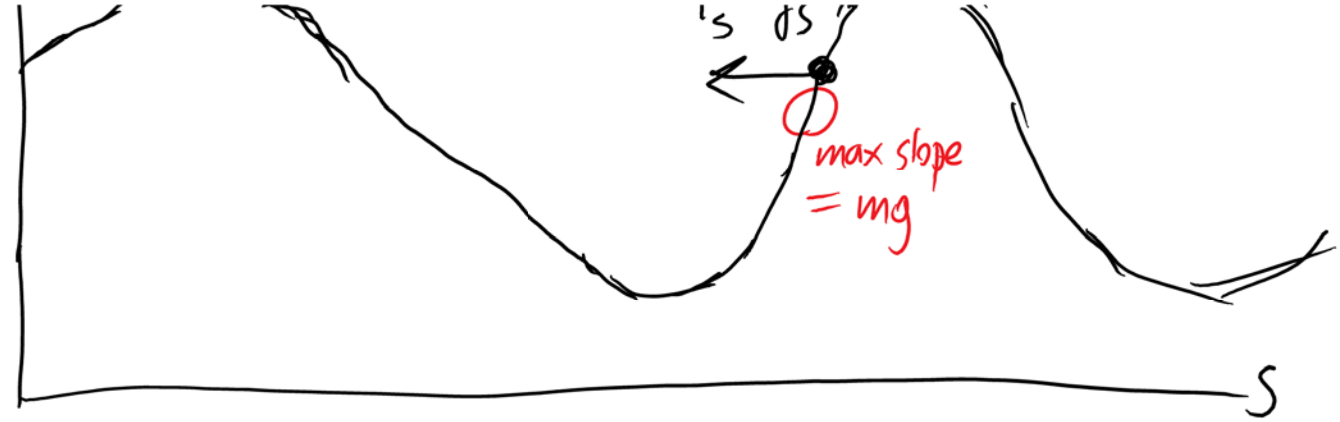
What about a roller coaster?



Motion of the roller coaster car is along the track, not along x.

For force from gradient, look at $U(s)$

$$F_s = \frac{-dU(s)}{ds}$$


max slope
= mg
S

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I'll post a pdf of today's lecture at
ter.ps/phys410-Sep04

Generalized coordinates

Let "q" be some arbitrary type of coordinate
(doesn't have to be a position)

Can talk about the generalized force, $-\dfrac{\partial U}{\partial q}$
"trying to change q"

e.g. torsion on a rod or wire

$$\Rightarrow \quad U = \frac{1}{2} K \theta^2$$

twist in radians

"spring constant" for rotation

Units of K: Joules (energy units)

Generalized force associated with rotation:

$$F_\theta = -\frac{\partial U}{\partial \theta} = -K\theta$$

units of F_θ: Energy units again (N·m or J)

This is actually torque

Useful things you can do with energy conservation:

- Get speed at some position, i.e. $v(x)$

$$E_{total} = K + U \quad \text{is conserved}$$

$$E = \frac{1}{2}m\dot{x}^2 + U(x) \Rightarrow \dot{x} = \sqrt{\frac{2(E-U(x))}{m}}$$

for whatever x

Tells you the speed there, but not the direction

- Get the elapsed time between two points

Separate vars and integrate

$$\int_{x_i}^{x_f} \frac{dx}{\sqrt{\frac{2}{m}(E-U(x))}} = \int_{t_i}^{t_f} dt = \Delta t$$

e.g. a mass on a spring

$$U = \frac{1}{2}kx^2$$

$$\Rightarrow \sqrt{\frac{m}{2}} \int_{x_i}^{x_f} \frac{dx}{\sqrt{E-\frac{1}{2}kx^2}}$$

let $\sqrt{\frac{k}{2}}x = \sqrt{E}y$

$\Rightarrow dx = \sqrt{\frac{2E}{k}} y$

$$= \sqrt{\frac{m}{2}}\sqrt{\frac{2E}{k}} \int \frac{dy}{\sqrt{E-Ey^2}}$$

$$= \sqrt{\frac{m}{k}} \int \frac{dy}{\ldots}$$

$$\underbrace{\int \sqrt{1-y^2}}_{\arcsin(y)}$$

$$= \sqrt{\frac{m}{k}} \arcsin\left(\sqrt{\frac{k}{2E}}\, x\right)\Bigg|_{x_0}^{x_f} = \Delta t$$

if $X_0 = 0$, then

$$\Delta t = \sqrt{\frac{m}{k}} \arcsin\left(\sqrt{\frac{k}{2E}}\, X_f\right)$$

$$\Rightarrow X_f = \underbrace{\sqrt{\frac{2E}{k}}}_{\text{Amplitude}} \sin\left(\sqrt{\frac{k}{m}}\, \Delta t\right)$$

ω for this system

Notes:
- You might have to integrate numerically
  - e.g. Mathematica : NIntegrate
  - Matlab: int
  - Excel, etc.

Riemann integration

- Assumes that the object doesn't change direction at a turning point — would have to break up the integral

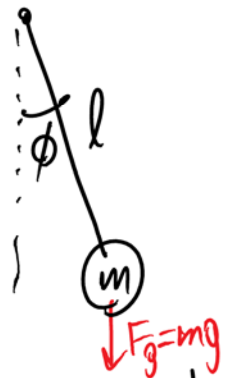
- Get the equation of motion

If $K + U = \text{constant}$, take deriv.

e.g. (simple) $\frac{1}{2}m\dot{x}^2 + U(x) = \text{const} \Rightarrow m\ddot{x} + \frac{dU}{dx} = 0$

$$\Rightarrow m\ddot{x} = -\frac{dU}{dx}$$

Another example:



$U(\phi) = mg \times \text{height} = mgl(1-\cos\phi)$ rel. to bottom of swing

$$\Rightarrow E = \frac{1}{2}m(l\dot{\phi})^2 + mgl(1-\cos\phi)$$

Take $\frac{d}{dt} \Rightarrow 0 = ml^2\dot{\phi}\ddot{\phi} + 0 + mgl\sin\phi\,\dot{\phi}$

$$= \dot{\phi}\,\underbrace{\left[ml^2\ddot{\phi} + mgl\sin\phi\right]}_{\text{must be zero}}$$

moment of inertia of the mass at the end of the string

neg. torque from gravity

must be zero — eqn of motion for ϕ (solve this diff. eqn. to get motion)

$$I\alpha = \tau_{net}$$

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What's conserved in a collision?

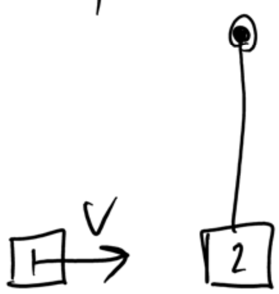
Mechanical energy is not always conserved

Momentum is conserved in a collision if the system is isolated, specifically: no net force on it

Mass — in a classical collision (not a relativistic one!)

Angular momentum is conserved in a collision if there's no net torque on the system

Example 1: Ballistic pendulum



Stick together when they collide

$\vec{L} = \vec{r} \times \vec{p}$
$(|\vec{L}| = r_\perp p)$

In this collision,
mech. energy — no — it's inelastic
linear mom. — yes (in x direction)
angular mom. — yes

Next time: Another example, and then Lagrangian mechanics